Teachers Dilemma

SETUP:

Teacher asks the student a question, of a certain difficulty, which the student answers correctly with some probability. Student gets points for getting the question correct and teacher gets points based on a combination of the difficulty of the question and if the student answers the question correctly. The goal is to structure the teacher payoff such that the teacher wants to pick a question that is in the students "zone of proximal development".

CHOICES AND ASSUMPTIONS:

1) The question difficulty will be taken as a real number between 0 and 1, where increasing values denote increasing difficulty.

2) The function $S(x)$ is the probability that a students gets a correct answer given a problem of difficulty x. The assumptions about $S(x)$ are:

a) $S(x): R \to R$
b) $S(x)$ is strictly decreasing
c) $S(x)$ is twice differentiable (-> continuous)
d) $S(0) > .5$ (possibly $S(0) = 1$)
e) $S(1) < .5$ (possibly $S(1) = 0$)

note: the output of $S(x)$ is a real number between 0 & 1, interpreted as the % chance of a successful answer by the student.

3) The student gets 1 point for a correct answer and 0 for incorrect.

4) The payoff the teacher receives is based on two linear functions. One linear function defines what the teacher gets if the student gets the question wrong. The other if the student gets it right.

Teacher payoff for student failure, $Tw(x) = xC + (1 - x)R$

Teacher payoff for student success, $Tc(x) = xJ + (1 - x)V$

where
x is the difficulty of the question.
C,R,J,V are constants > 0

We are allowed to vary these 4 parameters C,R,J,V to change the teacher's payoff function.

They are interpreted in the following way
C = Confirm (Hard question wrong)
R = Remediate (Easy question wrong)
J = Joy (Hard question correct)
V = Verify (Easy question correct)

5) Zone of proximal development is defined to be $S(x) = .5$. Which must occur because we defined $S(0) > .5$ and $S(1) < .5$.

6) The teachers payoff is defined to be the $T(x) = S(x)Tc(x) + (1-S(x))Tw(x)$.
i.e. (the probability that the student will give a correct answer) * (the payoff for a correct answer) + (the probability that the student will give a incorrect answer) * (the payoff for a incorrect answer)

PROBLEM:
Thus the problem becomes to select C,R,J,V so that the T(x) is at a max where $S(x) = .5$.

SOLUTION:
The discussion will be split into the general implications and specific solutions.

GENERAL IMPLICATIONS:

if $Tw(x)$ & $(Tc(x) - Tw(x))$ were decreasing and
$(Tc(x) - Tw(x))$ was > 0 then

$S(x)$ & $(Tc(x) - Tw(x))$ & $Tw(x)$ would all be positive decreasing functions.

so $S(x)(Tc(x) - Tw(x)) + Tw(x)$ (a different way to write $T(x)$) would also be a positive decreasing function. So $x=0$ would always give the maximum value for $T(x)$.

These requirements are met when

1) $Tw(x)$ is decreasing i.e. $C < R$

2) $(Tc(x) - Tw(x))$ is decreasing i.e. $(J - V) - (C - R) < 0$

3) and $(Tc(x) - Tw(x))$ is positive (on 0 to 1) i.e. $J - C > 0$ and $V - R > 0$

SPECIFIC SOLUTIONS:

First we'll do out T' and T''.

$$T'(x) = S(x)Tc'(x) + S'(x)Tr(x) - S(x)Tw'(x) - S'(x)Tw(x) + Tw'(x)$$

$$= S(x)(Tr'(x) - Tw'(x)) + S'(x)(Tr(x) - Tw(x)) + Tw'(x)$$

$$= S(x)(J - V - C + R) + S'(x)(xJ + (1-x)V - xC - (1-x)R) + C - R$$

$$= S(x)(J - V - C + R) + S'(x)(x(J - V - C + R) + V - R) + C - R$$

$$\begin{aligned} = \quad & S(x) (J - V - C + R) + \\ & S'(x) ((x (J - V - C + R)) + (V - R)) + \\ & C - R \end{aligned}$$

$$\begin{aligned} T''(x) = \quad & \\ & S'(x) (J - V - C + R) + \\ & S'(x) (J - V - C + R) + \end{aligned}$$

```
        S''(x) ( (x (J - V - C + R) ) + (V - R) ) +
        0
=       2*S'(x) (J - V - C + R) +
        S''(x) ( (x (J - V - C + R) ) + (V - R) )
```

Looking at various specific student functions

1) First try the simple linear function

```
        S(x) = 1 - x.
```

note : now T(x) can be written as

```
T(x) = (1 -x) (xJ + (1 - x)V) + (x) (xC + (1 - x)R)
= (1 -x) (xJ + V - xV) + (x) (xC + R - xR)
=(xJ + V - xV) + -(x^2J + xV - x^2V) + (x^2C + xR - x^2R)
= -x^2(J - V - C + R) + x(J - 2V + R) + V
```

a parabola.

Given this S(x) then

```
S(.5) = .5
S'(x) = -1
```

To find min/max values of T(x), we take T'(x) and set it to 0. To force .5 to be a max we set T'(.5) = 0. So we get

```
T'(x) = S(x) ( J - V - C + R) +
        S'(x) ( (x (J - V - C + R) ) + (V - R) ) +
        C - R

0 = .5 ( J - V - C + R) + -1 * ( ( .5 * ( J - V - C + R) ) + ( V - R ) ) +
C - R

0 = .5( J - V - C + R) + -.5( J - V - C + R) - (V - R) + C - R

0 =  -V + R + C - R

0 = C - V
```

So S(x) = .5 is a max or a min if C = V

Note: force this to be a max we need to make T''(x) < 0 we know

$S''(x) = 0$

calculating

$T''(x) = -2 (J - V - C + R)$

so $x=.5$ is a max if

$V + C < J + R$ or

$J - V > C - R$

but the end points could still be maximums

we know

$T(0) = V$

$T(1) = C$

$T(.5) = -.25(J - V - C + R) + .5(J - 2V + R) + V$
$= .25 J + .25V + .25 C + .25 R$
$> \max(V,C)$

but $V=C$

so

$= .25J + .25R + .5C$ want $> C$

since $J + R > V + C$ or $J + R > 2C$
then multiply by .25

$.25 J + .25 R > .5 C$

so $S(x) = .5$ is the max for $T(x)$

Conclusion
.5 is max if
1) $C=V$
2) $J + R > V + C$

2) Now a more complex function

$$S(x) = 1 / (1 + e^{(10x -5)})$$

note : now T(x) can be written as

$$T(x) = (1 / (1 + e^{(10x -5)})) (xJ + (1 - x)V) + (1 - (1 / (1 + e^{(10x -5)}))) (xC + (1 - x)R)$$

Given this S(x) then

$$S(.5) = .5$$
$$S'(x) = -(10 e^{(10x -5)}) / (1 + e^{(10x -5)})^2$$
$$S'(.5) = -2.5$$

To find min/max values of T(x), we take T'(x) and set it to 0. To force .5 to be a max we set T'(.5) = 0. So we get

$$\begin{aligned} T'(x) = {} & S(x) (J - V - C + R) + \\ & S'(x) ((x (J - V - C + R)) + (V - R)) + \\ & C - R \end{aligned}$$

$$\begin{aligned} 0 = {} & .5 (J - V - C + R) + \\ & -2.5 * ((.5 * (J - V - C + R)) + (V - R)) + \\ & C - R \end{aligned}$$

$$\begin{aligned} 0 = {} & .5(J - V - C + R) + \\ & -1.25(J - V - C + R) + -2.5(V - R) + \\ & C - R \end{aligned}$$

$$0 = -.75(J - V - C + R) + -2.5(V - R) + C - R$$

$$0 = -.75J + -1.75V +1.75C +.75R$$

$$0 = -3J -7V +7C +3R$$

$$7C + 3R = 3J + 7V$$

So S(x) = .5 is a max or a min if 7C + 3R = 3J + 7V

Note: force this to be a max we need to make T''(x) < 0

we know

S''(x) = -10(

((10e^(10x-5)) / ((1 + e^(10x-5)) ^2))

- ((20e^(20x-10)) / ((1 + e^(10x-5)) ^ 3))

)

S''(.5) = -10(

((10) / ((1 + 1) ^2))

- ((20) / ((1 + 1) ^ 3))

)

= -10(10/4 - 20/8) = 0

calculating

```
T''(x)
=       2*S'(x) (J - V - C + R) +
        S''(x) ( (x (J - V - C + R) ) + (V - R) )
```

T''(.5) = 2 * -2.5 (J - V - C + R) = -5 (J - V - C + R)

so x=.5 is a max if

V + C < J + R or

J - V > C - R

TODO: must eliminate other Maxima possibilities (end points etc...) but the end points could still be greater maximums

Conclusion

.5 is max if

1) $7C + 3R = 3J + 7V$

2) $J + R > V + C$